

HIGHER TOPOLOGICAL COMPLEXITY OF ASPHERICAL SPACES

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ABSTRACT. In this article we study the higher topological complexity $\mathrm{TC}_r(X)$ in the case when X is an aspherical space, $X = K(\pi, 1)$ and $r \geq 2$. We give a characterisation of $\mathrm{TC}_r(K(\pi, 1))$ in terms of classifying spaces for equivariant Bredon cohomology. Our recent paper [8], joint with M. Grant and G. Lupton, treats the special case $r = 2$. We also obtain in this paper useful lower bounds for $\mathrm{TC}_r(\pi)$ in terms of cohomological dimension of subgroups of $\pi \times \pi \times \cdots \times \pi$ (r times) with certain properties. As an illustration of the main technique we find the higher topological complexity of the Higman's groups. We also apply our method to obtain a lower bound for the higher topological complexity of the right angled Artin (RAA) groups, which, as was established in [17] by a different method (in a more general situation), coincides with the precise value. We finish the paper by a discussion of the TC-generating function $\sum_{r=1}^{\infty} \mathrm{TC}_{r+1}(X)x^r$ encoding the values of the higher topological complexity $\mathrm{TC}_r(X)$ for all values of r . We show that in many examples (including the case when $X = K(H, 1)$ with H being a RAA group) the TC-generating function is a rational function of the form $\frac{P(x)}{(1-x)^2}$ where $P(x)$ is an integer polynomial with $P(1) = \mathrm{cat}(X)$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

1.1. Suppose that a mechanical system has to be programmed to move autonomously from any initial state to any final state. Let X denote the configuration space of the system; points of X represent states of the system and continuous paths in X represent motions of the system. A *motion planning algorithm* is a function which associates with any pair of states $(A, B) \in X \times X$ a continuous motion of the system starting at A and ending at B . In other words, a motion planning algorithm is a section of the path fibration

$$(1) \quad p : X^I \rightarrow X \times X, \quad p(\gamma) = (\gamma(0), \gamma(1)).$$

Here X^I denotes the space of all continuous paths $\gamma : I = [0, 1] \rightarrow X$, equipped with the compact-open topology. Unfortunately, a global motion planning algorithm is impossible to achieve unless the configuration space X is contractible. If X is not contractible, then only “local” motion plans may be found.

The *topological complexity*, $\mathrm{TC}(X)$, informally, is the minimal number of continuous rules (i.e. local planners) which are needed to construct an algorithm for autonomous motion planning of a system having X as its configuration space. The quantity $\mathrm{TC}(X)$, originally introduced in [6] (see also [10]), is, in fact, a numerical homotopy invariant of a path-connected topological space X and so may be studied with all the tools of algebraic

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topology. A recent survey of the concept $\mathrm{TC}(X)$ and robot motion planning algorithms in practically interesting configuration spaces can be found in [11].

1.2. The concept of higher or sequential topological complexity. Yuli Rudyak [27] introduced a generalisation of the notion of topological complexity $\mathrm{TC}(X)$ which is usually denoted $\mathrm{TC}_r(X)$ and is called *the higher* or *sequential* topological complexity. Here X is a path-connected topological space and $r \geq 2$ is an integer. The number $\mathrm{TC}_2(X)$ coincides with $\mathrm{TC}(X)$. To define $\mathrm{TC}_r(X)$ consider the fibration

$$(2) \quad p_r : X^I \rightarrow X^r,$$

where

$$(3) \quad p_r(\gamma) = (\gamma(0), \gamma\left(\frac{1}{r-1}\right), \gamma\left(\frac{2}{r-1}\right), \dots, \gamma\left(\frac{r-2}{r-1}\right), \gamma(1)), \quad \gamma \in X^I.$$

As above, X^I denotes the space of all continuous paths $\gamma : I = [0, 1] \rightarrow X$ equipped with the compact-open topology. The notation X^r denotes $X \times X \times \dots \times X$ (r times), the Cartesian product of r copies of X . The map p_r associates with a path $\gamma : I \rightarrow X$ in X the sequence of its locations at r points $\frac{i}{r-1} \in I$ where $i = 0, 1, \dots, r-1$ includes the initial and final states $\gamma(0), \gamma(1)$ and $r-2$ intermediate points.

Definition 1.1. *Given a path-connected topological space X , the r -th sequential topological complexity of X is defined as the minimal integer $\mathrm{TC}_r(X) = k$ such that the Cartesian power X^r can be covered by $k+1$ open subsets*

$$X^r = U_0 \cup U_1 \cup \dots \cup U_k$$

with the property that for any $i = 0, 1, 2, \dots, k$ there exists a continuous section

$$s_i : U_i \rightarrow X^I, \quad p_r \circ s_i = \mathrm{id}$$

of the fibration (2) over U_i . If no such k exists we will set $\mathrm{TC}_r(X) = \infty$.

In other words, $\mathrm{TC}_r(X)$ is the Schwarz genus (or sectional category) of fibration (2), see [28].

1.3. The invariant $\mathrm{TC}_r(X)$ has a clear meaning for the motion planning problem of robotics. Assume that a system (robot) has to be programmed to move autonomously from any initial state to any final state such that it visits $r-2$ additional states on the way. If X denotes the configuration space of the system then $\mathrm{TC}_r(X)$ is the minimal number of continuous rules needed to program the robot to perform autonomously the indicated task. We note here that the most basic estimate for TC_r is in terms of the Lusternik-Schnirelmann category (see [3]):

$$(4) \quad \mathrm{cat}(X^{r-1}) \leq \mathrm{TC}_r(X) \leq \mathrm{cat}(X^r) \leq r \mathrm{cat}(X).$$

We won't require any results about category here except for the fact that $\mathrm{cat}(K(\pi, 1)) = \mathrm{cd}(\pi)$, where $\mathrm{cd}(\pi)$ is the cohomological dimension of the group π . Here the symbol $K(\pi, 1)$ denotes the Eilenberg - MacLane space with the properties $\pi_i(K(\pi, 1)) = 0$ for $i \neq 1$ and $\pi_1(K(\pi, 1)) = \pi$. Some recent results concerning the invariant $\mathrm{TC}_r(X)$ with $r \geq 2$ can be

found in [1], [15], [17], [30]. Also, the introduction to [8] gives an account of most of the significant recent developments regarding $\mathrm{TC}(X)$.

1.4. One of the main properties of $\mathrm{TC}_r(X)$ is its homotopy invariance, which means that $\mathrm{TC}_r(X) = \mathrm{TC}_r(Y)$ if X and Y are homotopy equivalent. In particular we obtain that in the case when $X = K(\pi, 1)$ the number $\mathrm{TC}_r(X)$ is an algebraic invariant of the group π . We shall introduce the notation

$$\mathrm{TC}_r(\pi) = \mathrm{TC}_r(K(\pi, 1)).$$

Our aim in this paper is to give a characterisation of $\mathrm{TC}_r(\pi)$ in terms of equivariant topology. On the face of it, there seems to be no connection between these topics, but in our main result Theorem 3.1 we will describe a path that unites them. In fact, Theorem 3.1 generalizes a similar connection that was displayed in [8] for TC itself. The emergence of equivariant topology as a player in the study of TC allows invariants of the subject such as Bredon cohomology (with respect to a family of subgroups) to be used to estimate TC . Moreover, as a result of Theorem 3.1, new and interesting lower bounds are obtained for TC_r as in Theorem 2.1 below. In previous works, lower bounds for TC_r tended to arise from cohomology (as “cuplength”-type calculations). Here, however, Theorem 2.1 gives a lower bound that is more intrinsic to the subgroup structure of π . This type of result can be applied even when cuplength structure is missing (as in Theorem 2.2 below).

2. A LOWER BOUND FOR $\mathrm{TC}_r(\pi)$

2.1. In this subsection we shall state a useful corollary of our main result (Theorem 3.1) which gives a lower bound for $\mathrm{TC}_r(\pi)$; it has the advantage of being stated using very simple algebraic terms.

Fix an integer $r \geq 2$ and consider the r -fold Cartesian product

$$\pi^r = \pi \times \pi \times \cdots \times \pi.$$

We shall consider the diagonal subgroup

$$(5) \quad \Delta = \{(g, g, \dots, g); g \in \pi\} \subset \pi^r.$$

Any subgroup $H \subset \pi^r$ conjugate to the diagonal Δ has the form $H = c^{-1}\Delta c$ where $c = (c_1, c_2, \dots, c_r) \in \pi^r$. Consider a subgroup $K \subset \pi^r$ with the property that $K \cap H = \{1\}$ for any subgroup $H \subset \pi^r$ conjugate to Δ . This property of K can be characterised as follows. For an element $g \in \pi$ we shall denote by $[g] \subset \pi$ its conjugacy class. Then for any non-unit element $1 \neq (g_1, g_2, \dots, g_r) \in K$ there exists $i, j \in \{1, 2, \dots, r\}$ such that $[g_i] \neq [g_j]$.

Theorem 2.1. *Let π be a discrete group, $r \geq 2$ an integer, and let $K \subset \pi^r = \pi \times \pi \cdots \times \pi$ be a subgroup with the property that $K \cap H = 1$ for any subgroup $H \subset \pi^r$ conjugate to the diagonal $\Delta \subset \pi^r$. Then*

$$(6) \quad \mathrm{TC}_r(\pi) \geq \mathrm{cd}(K),$$

where $\mathrm{cd}(K)$ denotes the cohomological dimension of K .

Theorem 2.1 generalises Corollary 3.5.4 from [8] (where the case $r = 2$ was covered) as well as the result of [18] where the class of subgroups of type $K = A \times B$ is considered assuming that $r = 2$.

Theorem 2.1 may be applied in the following situations. In the special case when the subgroup $K \subset \pi^r$ has the form $K = A_1 \times A_2 \times \cdots \times A_r$ with $A_i \subset \pi$, the assumption of Theorem 2.1 requires that for any collection $g_1, \dots, g_r \in \pi$ the intersection

$$\bigcap_{i=1}^r g_i A_i g_i^{-1} = \{1\}$$

is trivial. In particular, we may take $K = A \times B \times \pi \times \cdots \times \pi$ with the subgroups $A, B \subset \pi$ satisfying $A \cap gBg^{-1} = \{1\}$ for any $g \in \pi$ as in [18].

One may always apply Theorem 2.1 with $K = 1 \times \pi^{r-1} \subset \pi^r$ which gives the well-known inequality

$$\mathrm{TC}_r(\pi) \geq \mathrm{cd}(\pi^{r-1}).$$

For the free abelian group $\pi = \mathbb{Z}^k$ one has $\mathrm{TC}_r(\pi) = (r-1)k = \mathrm{cd}(\pi^{r-1})$, i.e. in this case the above inequality is sharp.

We shall use Theorem 2.1 to prove the following:

Theorem 2.2. *Let \mathcal{H} denote Higman's group with presentation*

$$P = \langle x, y, z, w \mid xyx^{-1}y^{-2}, yzy^{-1}z^{-2}, zwz^{-1}w^{-2}, wxw^{-1}x^{-2} \rangle.$$

Then

$$\mathrm{TC}_r(\mathcal{H}) = 2r \quad \text{for any } r \geq 2.$$

The proof of Theorem 2.1 is given in §3.4. The proof of Theorem 2.2 is given in §6.

3. STATEMENT OF THE MAIN RESULT

3.1. To state the main result we need to recall a few standard notions; we shall mainly follow [22].

Let G be a discrete group. A G -CW-complex is a CW-complex X with a left G -action such that for each open cell $e \subset X$ and each $g \in G$ with $ge \cap e \neq \emptyset$, the left multiplication by g acts identically on e .

A family \mathcal{F} of subgroups of G is a set of subgroups of G which is closed under conjugation and finite intersections.

3.2. A classifying G -CW-complex $E_{\mathcal{F}}(G)$ with respect to a family \mathcal{F} is defined as a G -CW-complex $E_{\mathcal{F}}(G)$ such that

- (a) the isotropy subgroup of any element of $E_{\mathcal{F}}(G)$ belongs to \mathcal{F} ;
- (b) For any G -CW-complex Y whose all isotropy subgroups belong to \mathcal{F} there is up to G -homotopy exactly one G -map $Y \rightarrow E_{\mathcal{F}}(G)$.

A G -CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy subgroups belong to the family \mathcal{F} and for each $H \in \mathcal{F}$ the set of H -fixed points X^H is weakly contractible, i.e. $\pi_i(X^H, x_0) = 0$ for any $i = 0, 1, \dots$ and for any $x_0 \in X^H$. See [22], Theorem 1.9.

3.3. We shall assume below that a discrete group π is fixed. Let G denote the group π^r where $r \geq 2$. Let $\Delta \subset \pi^r$ denotes the diagonal subgroup (5). Let \mathcal{D} be the minimal family of subgroups of G containing the diagonal subgroup $\Delta \subset \pi^r$ and the trivial subgroup, which is closed under conjugations by elements of π^r and under taking finite intersections.

We shall consider the classifying spaces $E(G)$ and $E_{\mathcal{D}}(G)$, where $E(G)$ is the classical classifying space for free actions. The universal properties of classifying spaces imply the existence of a G -map

$$(7) \quad f : E(G) \rightarrow E_{\mathcal{D}}(G),$$

which is unique up to equivariant homotopy.

Now we may state the main result of this paper:

Theorem 3.1. *Let X be a finite aspherical cell complex, let $\pi = \pi_1(X, x_0)$ be its fundamental group and let $r \geq 2$ be an integer. Denote $G = \pi^r = \pi \times \cdots \times \pi$. Then the topological complexity $\mathrm{TC}_r(X) = \mathrm{TC}_r(\pi)$ coincides with the smallest integer k such that the canonical map (7) can be factorised (up to G -equivariant homotopy) as*

$$(8) \quad E(G) \rightarrow L \rightarrow E_{\mathcal{D}}(G)$$

where L is a G -CW-complex of dimension $\leq k$. Here \mathcal{D} is the family of subgroups of G defined above. Equivalently, $\mathrm{TC}_r(X)$ coincides with the smallest integer k such that the canonical map (7) is G -equivariantly homotopic to a map taking values in the k -dimensional skeleton

$$E_{\mathcal{D}}(G)^{(k)} \subset E_{\mathcal{D}}(G).$$

Theorem 3.1 and its proof generalise the results of [8] where the case $r = 2$ was treated.

It is known that the classifying space $E_{\mathcal{D}}(G)$ admits a realisation as a G -CW-complex of dimension $\max\{3, \mathrm{cd}_{\mathcal{D}}(G)\}$, see [22]. Here the symbol $\mathrm{cd}_{\mathcal{D}}(G)$ stands for the cohomological dimension of the trivial $O_{\mathcal{D}}$ -module \mathbb{Z} and $O_{\mathcal{D}}$ denotes the orbit category with orbits of type \mathcal{D} , see [22] or [26]. Hence we obtain the following corollary:

Corollary 3.2. *One has*

$$(9) \quad \mathrm{TC}_r(\pi) \leq \max\{3, \mathrm{cd}_{\mathcal{D}}(\pi^r)\}, \quad r \geq 2.$$

Later in this paper we shall supplement this upper bound on $\mathrm{TC}_r(\pi)$ by lower bounds based on Bredon cohomology.

3.4. Proof of Theorem 2.1. In this subsection we show how Theorem 2.1 follows from Theorem 3.1. Denote $G = \pi^r$ and consider a decomposition

$$E(G) \xrightarrow{\alpha} L \xrightarrow{\beta} E_{\mathcal{D}}(G),$$

where L is a G -CW-complex of dimension $\dim L = \mathrm{TC}_r(\pi)$, as given by Theorem 3.1; here α and β are G -equivariant maps. The subgroup $K \subset G$ acts freely on $E(G)$ and on $E_{\mathcal{D}}(G)$ - here we use our assumption on the subgroup K . Hence both spaces $E(G)$ and $E_{\mathcal{D}}(G)$ can be viewed as models of $E(K)$. We may fix K -equivariant homotopy equivalences

$a : E(K) \rightarrow E(G)$ and $b : E(K) \rightarrow E_{\mathcal{D}}(G)$. For any G -equivariant map $f : E(G) \rightarrow E_{\mathcal{D}}(G)$ (as (7)) we have

$$f \circ a \simeq_K b$$

as follows from the universal property of $E(K)$; here the sign \simeq_K denotes a K -equivariant homotopy. Taking $f = \beta \circ \alpha$, we see that for any $\mathbb{Z}[K]$ -module M and for any cohomology class $\gamma \in H^i(E(K), M) = H^i(K, M)$, with

$$i > \mathrm{TC}_r(\pi) = \dim L,$$

we may write $\gamma = b^*(\gamma')$ where $\gamma' \in H^i(E_{\mathcal{D}}(G), M)$ and for obvious reasons the class $\alpha^*\beta^*(\gamma') \in H^i(E(G), M)$ is trivial. This implies that

$$\gamma = b^*(\gamma') = a^*f^*(\gamma') = a^*\alpha^*\beta^*(\gamma') = 0,$$

i.e. $H^i(K, M) = 0$ for any $i > \mathrm{TC}_r(\pi)$ and for any $\mathbb{Z}[K]$ -module M . This proves that $\mathrm{cd}(K) \leq \mathrm{TC}_r(\pi)$ as claimed. \square

4. PROOF OF THEOREM 3.1

4.1. The invariant $\mathrm{TC}_r^{\mathcal{D}}(X)$. We shall use a convenient modification of the concept $\mathrm{TC}_r(X)$. As before we denote by \mathcal{D} the family of subgroups of $G = \pi^r$ generated by the diagonal Δ .

Definition 4.1. *Let X be a path-connected topological space with fundamental group $\pi = \pi_1(X, x_0)$ and $r \geq 2$ an integer. The \mathcal{D} -topological complexity, $\mathrm{TC}_r^{\mathcal{D}}(X)$, is defined as the minimal number k such that X^r can be covered by $k + 1$ open subsets*

$$X^r = U_0 \cup U_1 \cup \dots \cup U_k$$

with the property that for any $i = 0, 1, 2, \dots, k$ and for any choice of the base point $u_i \in U_i$ the homomorphism $\pi_1(U_i, u_i) \rightarrow \pi_1(X^r, u_i)$ induced by the inclusion $U_i \rightarrow X^r$ takes values in a subgroup conjugate to the diagonal $\Delta \subset \pi^r$.

To ensure that Definition 4.1 makes sense, recall that for any choice of the base point $u = (u_1, \dots, u_r) \in X^r$ one has

$$\pi_1(X^r, u) = \pi_1(X, u_1) \times \pi_1(X, u_2) \times \dots \times \pi_1(X, u_r)$$

and there is an isomorphism $\pi_1(X^r, u) \rightarrow \pi_1(X^r, (x_0, \dots, x_0)) = \pi^r$ determined uniquely up to conjugation. Moreover, the diagonal inclusion $X \rightarrow X^r$, $x \mapsto (x, x, \dots, x)$, induces the inclusion $\pi \rightarrow \pi^r$ onto the diagonal Δ .

Lemma 4.2. *If X is a finite aspherical cell complex then $\mathrm{TC}_r^{\mathcal{D}}(X) = \mathrm{TC}_r(X)$.*

Proof. Consider an open subset $U \subset X^r$ and a continuous section $s : U \rightarrow X^I$ of the fibration (2) over U . Using the exponential correspondence, the map s can be viewed as a homotopy $h : U \times I \rightarrow X$ where $h(x, t) = s(x)(t)$ for $x \in U, t \in I$. One has

$$h\left(x, \frac{i-1}{r-1}\right) = x_i \quad \text{for } i = 1, \dots, r$$

where $x = (x_1, \dots, x_r)$. Let $q_j : X^r \rightarrow X$ (where $j = 1, \dots, r$) denote the projection onto the j -th factor. The property of s to be a section of (2) can be expressed by saying that the homotopy $h|U \times [\frac{i-1}{r}, \frac{i}{r}]$ connects the projections $q_i : U \rightarrow X$ and $q_{i+1} : U \rightarrow X$ for $j = 1, \dots, r$.

Thus we see that the open sets $U_i \subset X \times X$ which appear in Definition 1.1 can be equivalently characterised by the property that their projections $q_j : U_i \rightarrow X$ on all the factors $j = 1, \dots, r$ are homotopic to each other.

Since X is aspherical, for any connected space U , which is homotopy equivalent to a cell complex, the set of homotopy classes of maps $U \rightarrow X$ is in one-to-one correspondence with the set of conjugacy classes of homomorphisms $\pi_1(U, u) \rightarrow \pi_1(X, x_0)$, see [31], Chapter V, Corollary 4.4. Recall that an open subset of a CW-complex is an ANR and therefore is homotopy equivalent to a countable CW-complex, see Theorem 1 in [25]. Thus we see that an open subset $U \subset X^r$ admits a continuous section of fibration (1) if and only if the induced homomorphisms

$$q_{j*} : \pi_1(U_1, u) \rightarrow (X, u_j), \quad j = 1, \dots, r$$

are conjugate to each other. Here $u = (u_1, \dots, u_r) \in U$ is a base point. This latter condition is obviously equivalent to the requirement (which appears in Definition 4.1) that the map on π_1 induced by the inclusion $U \rightarrow X^r$ takes values in a subgroup conjugate to the diagonal Δ . This completes the proof. \square

Corollary 4.3. *Let X be a connected finite aspherical cell complex with fundamental group $\pi = \pi_1(X, x_0)$ and $r \geq 2$. Let $q : \widehat{X}^r \rightarrow X^r$ be the connected covering space corresponding to the diagonal subgroup*

$$\Delta \subset \pi^r = \pi_1(X^r, (x_0, \dots, x_0)).$$

Then the \mathcal{D} -topological complexity $\mathrm{TC}_r^{\mathcal{D}}(X)$ coincides with the Schwarz genus of q .

Proof. For an open subset $U \subset X^r$, the condition that the induced map $\pi_1(U, u) \rightarrow \pi_1(X^r, u)$ takes values in a subgroup conjugate to the diagonal Δ is equivalent to the condition that q admits a continuous section over U . The Lemma follows by comparing the definitions of $\mathrm{TC}_r^{\mathcal{D}}(X)$ and of Schwarz genus. \square

4.2. The covering $q : \widehat{X}^r \rightarrow X^r$ which appears in Corollary 4.3 is a regular covering only when π is abelian. This covering can be characterised by the property that the image of the homomorphism $q_* : \pi_1(\widehat{X}^r) \rightarrow \pi_1(X^r)$ is a subgroup conjugate to $\Delta \subset \pi^r$. Below we describe this covering in more detail.

4.3. Let π be a discrete group and $r \geq 2$. Denote $G = \pi^r$. We shall view

$$\pi^{r-1} = \pi \times \dots \times \pi$$

(the Cartesian product of $r-1$ copies of π) as a discrete topological space with the following left $G = \pi^r$ -action:

$$(10) \quad (x_0, x_1, \dots, x_r) \cdot (g_1, \dots, g_r) = (x_0 g_1 x_1^{-1}, x_1 g_2 x_2^{-1}, \dots, x_{r-1} g_r x_r^{-1}),$$

where $(x_0, \dots, x_r) \in \pi^r$ and $(g_1, \dots, g_r) \in \pi^{r-1}$. This action is transitive and the isotropy subgroup of the element $(1, 1, \dots, 1) \in \pi^{r-1}$ coincides with the diagonal subgroup $\Delta \subset \pi^r = G$. The isotropy subgroups of the other elements are the conjugates of Δ .

Consider the universal covering $\tilde{X}^r \rightarrow X^r$. The space \tilde{X}^r carries a free left G -action and we may consider $\tilde{X}^r \rightarrow X^r$ as a principal G -fibration. The associated fibration

$$(11) \quad \tilde{X}^r \times_G \pi^{r-1} \rightarrow X^r$$

(the Borel construction) coincides with the covering $q : \widehat{X}^r \rightarrow X^r$. Indeed, since

$$\tilde{X}^r \times_G \pi^{r-1} = \tilde{X}^r \times_G (G/\Delta) = \tilde{X}^r/\Delta,$$

the fundamental group of the space $\tilde{X}^r \times_G \pi^{r-1}$ can be naturally identified with Δ and therefore (11) coincides with the covering corresponding to Δ . Thus, $\mathrm{TC}_r^{\mathcal{D}}(X)$ coincides with the Schwarz genus of the fibration (11).

4.4. The join $X * Y$ of topological spaces X and Y can be defined as the quotient of the product $X \times [0, 1] \times Y$ with respect to the equivalence relation $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$ for all $x, x' \in X$ and $y, y' \in Y$. We have an obvious embedding $X \rightarrow X * Y$ given by $x \mapsto (x, 0, y)$ where $y \in Y$ is arbitrary.

A point $(x, t, y) \in X \times [0, 1] \times Y / \sim$ can be written as a formal linear combination $(1-t)x + ty$. This notation is clearly consistent with the identifications of the join.

4.5. For an integer $k \geq 0$, let $E_k(\pi^{r-1})$ denote the $(k+1)$ -fold join

$$E_k(\pi^{r-1}) = \pi^{r-1} * \pi^{r-1} * \dots * \pi^{r-1}.$$

We shall equip $E_k(\pi^{r-1})$ with the left diagonal $G = \pi^r$ -action determined by the G -action on π^{r-1} , see (10). Each $E_k(\pi^{r-1})$ is naturally a k -dimensional equivariant simplicial complex with k -dimensional simplexes in 1-1 correspondence with sequences (g_0, g_1, \dots, g_k) of elements $g_i \in \pi^{r-1}$. Note also that $E_k(\pi^{r-1})$ is $(k-1)$ -connected.

4.6. Next we apply a theorem of A. Schwarz (see [28], Theorem 3) stating that genus of a fibration $p : E \rightarrow B$ equals the smallest integer k such that the fiberwise join $p * p * \dots * p$ of $k+1$ copies of $p : E \rightarrow B$ admits a continuous section. We apply this criterion to the fibration (11). The fiberwise join of $k+1$ copies of (11) is obviously the fibration

$$(12) \quad q_k : \tilde{X}^r \times_G E_k(\pi^{r-1}) \rightarrow X^r,$$

where the left G -action on $E_k(\pi^{r-1})$ is described above. Hence we obtain that the number $\mathrm{TC}_r^{\mathcal{D}}(X)$ coincides with the smallest k such that (12) admits a continuous section.

4.7. Finally we apply Theorem 8.1 from [21], chapter 4, which states that continuous sections of the fibre bundle q_k are in 1-1 correspondence with G -equivariant maps

$$(13) \quad \tilde{X}^r \rightarrow E_k(\pi^{r-1}).$$

Thus, we see that $\mathrm{TC}_r^{\mathcal{D}}(X)$ is the smallest k such that a G -equivariant map (13) exists.

4.8. In the rest of the proof we shall assume that X is an aspherical finite cell complex. We observe that the space of the universal cover \tilde{X}^r is a contractible CW-complex with a free G -action, thus \tilde{X}^r is a model of the classifying space $E(G)$.

4.9. There is a natural equivariant embedding

$$E_k(\pi^{r-1}) \hookrightarrow E_{k+1}(\pi^{r-1}) = E_k(\pi^{r-1}) * \pi^{r-1}.$$

Using it we may define a G -CW-complex

$$E(\pi^{r-1}) = \bigcup_{k=1}^{\infty} E_k(\pi^{r-1}) = \pi^{r-1} * \pi^{r-1} * \pi^{r-1} * \dots,$$

the join of infinitely many copies of π^{r-1} . We claim that the G -complex $E(\pi^{r-1})$ is a model for the classifying space $E_{\mathcal{D}}(G)$. Indeed, $E(\pi^{r-1})$ is a simplicial complex with a simplicial G -action hence a G -CW-complex (with respect to the barycentric subdivision), see [22], Example 1.5.

We want to show that: (a) the isotropy subgroup of every point $x \in E(\pi^{r-1})$ belongs to the family \mathcal{D} and (b) that for any $H \in \mathcal{D}$ the fixed point set $E(\pi^{r-1})^H$ is contractible. Any point $x \in E(\pi^{r-1})$ can be represented in the form

$$x = t_0 x_0 + t_1 x_1 + \dots + t_k x_k$$

where $t_i \in (0, 1]$, $x_i \in \pi^{r-1}$ and $t_0 + t_1 + \dots + t_k = 1$. Then the isotropy subgroup of x is the intersection of the isotropy subgroups of $x_i \in \pi^{r-1}$ which are all conjugates of Δ ; thus the isotropy subgroup of x is a member of the family \mathcal{D} . If $H \in \mathcal{D}$ then the set $E(\pi^{r-1})^H$ coincides with the infinite join

$$(\pi^{r-1})^H * (\pi^{r-1})^H * (\pi^{r-1})^H * \dots$$

which is obviously contractible. We see that properties (a) and (b) are satisfied and therefore the space $E(\pi^{r-1})$ is a model of the classifying space $E_{\mathcal{D}}(G)$.

4.10. Now we shall use the main properties of classifying spaces $E(G)$ and $E_{\mathcal{D}}(G)$, see §3.2. In particular any G -CW-complex Y with isotropy in class \mathcal{D} admits a unique up to G -homotopy G -map $Y \rightarrow E_{\mathcal{D}}(G)$. In particular, there exists unique up to homotopy maps

$$f : E(G) \rightarrow E_{\mathcal{D}}(G), \quad \text{and} \quad g : E_k(\pi^{r-1}) \rightarrow E_{\mathcal{D}}(G).$$

One option for g is the natural inclusion $E_k(\pi^{r-1}) \rightarrow E(\pi^{r-1}) = E_{\mathcal{D}}(G)$. The map f can be realised as follows. Let $\phi : \pi^r \rightarrow \pi^{r-1}$ be given by

$$(14) \quad \phi(x_0, x_1, \dots, x_r) = (y_1, y_2, \dots, y_r), \quad \text{where} \quad y_i = x_{i-1} x_i^{-1}$$

where for $i = 1, \dots, r$. It is easy to see that ϕ is G -equivariant. The natural extension of ϕ to the infinite joins defines a G -equivariant map

$$f : E(G) = \pi^r * \pi^r * \pi^r * \dots \rightarrow E_{\mathcal{D}}(G) = E(\pi^{r-1}) = \pi^{r-1} * \pi^{r-1} * \pi^{r-1} * \dots$$

4.11. We have shown above (see Lemma 4.2 and §4.7) that $\mathrm{TC}_r(\pi)$ coincides with the smallest k such that there exists a G -equivariant map $h : E(G) \rightarrow E_k(\pi^{r-1})$. Composing with g (see above) we obtain a G -map $g \circ h : E(G) \rightarrow E_{\mathcal{D}}(G)$ which must be G -homotopic to f . We see that for $k = \mathrm{TC}_r(\pi)$ the map (7) factorises (up to G -homotopy) as

$$E(G) \rightarrow E_k(\pi^{r-1}) \rightarrow E_{\mathcal{D}}(G),$$

where $\dim E_k(\pi^{r-1}) = k$.

On the other hand, suppose that the map (7) factorises as follows

$$E(G) = E(\pi^r) \xrightarrow{\alpha} L \xrightarrow{\beta} E_{\mathcal{D}}(G) = E(\pi^{r-1})$$

with $\dim L \leq d$. We want to apply the equivariant Whitehead Theorem (see Theorem 4.4 below) to the inclusion $E_d(\pi^{r-1}) \rightarrow E(\pi^{r-1})$. For any subgroup $H \subset G$ we have

$$E_d(\pi^{r-1})^H = (\pi^{r-1})^H * (\pi^{r-1})^H * \dots * (\pi^{r-1})^H,$$

with $d+1$ factors. Thus, $E_d(\pi^{r-1})^H$ is $d-1$ -connected. Besides, $E(\pi^{r-1})^H$ is contractible. The Whitehead Theorem applied to $\beta : L \rightarrow E(\pi^{r-1})$ gives a G -map $\gamma : L \rightarrow E_d(\pi^{r-1})$. Composing with α we obtain $\gamma \circ \alpha : E(G) \rightarrow E_d(\pi^{r-1})$; thus, using the results of Lemma 4.2 and §4.7, we obtain that $\mathrm{TC}_r(\pi) \leq d$. This completes the proof. \square

Theorem 4.4 (Whitehead theorem, see [23], Theorem 3.2 in Chapter 1). *Let $f : Y \rightarrow Z$ be a G -map between G -CW-complexes such that for each subgroup $H \subset G$ the induced map $\pi_i(Y^H, x_0) \rightarrow \pi_i(Z^H, f(x_0))$ is an isomorphism for $i < k$ and an epimorphism for $i = k$ for any base point $x_0 \in Y^H$. Then for any G -CW-complex L the induced map on the set of G -homotopy classes*

$$f_* : [L, Y]_G \rightarrow [L, Z]_G$$

is an isomorphism if $\dim L < k$ and an epimorphism if $\dim L \leq k$.

5. LOWER BOUNDS FOR $\mathrm{TC}_r(\pi)$ VIA BREDON COHOMOLOGY

In the theory of Lusternik - Schnirelmann category the following result plays an important role. If $X = K(\pi, 1)$ is an aspherical space and $H^n(X; M) \neq 0$ for some local coefficient system then $\mathrm{cat}(X) \geq n$, see [5], [28]. A word-to-word generalisation of this result for $\mathrm{TC}(X)$ fails as we have many examples of aspherical spaces $X = K(\pi, 1)$ such that $H^n(X \times X, M) \neq 0$ while $\mathrm{TC}(X) < n$.

In [12] a notion of *an essential cohomology class* was introduced and the existence of a nonzero essential class $\xi \in H^n(\pi \times \pi, M)$ implies $\mathrm{TC}(\pi) \geq n$.

In this paper we generalise the approach of [8] of using Bredon cohomology to detect essential cohomology classes. Namely we show that for any $r \geq 2$ the existence of a nonzero cohomology class $\xi \in H^n(\pi^r, M)$ which can be extended to a Bredon cohomology class $\xi \in H_{\mathcal{D}}^n(\pi^r, \underline{M})$ (with respect to the family \mathcal{D} of subgroups of π^r which was described earlier) implies that $\mathrm{TC}_r(\pi) \geq n$. For $r = 2$ this was proven in [8].

5.1. Bredon cohomology. First we recall the construction of Bredon cohomology, see for example [26].

As above, let G denote the group π^r , where $r \geq 2$, and \mathcal{D} denote the minimal family of subgroups of π^r containing the diagonal $\Delta \subset \pi^r = G$ and the trivial subgroup $\{1\} \subset G$ which is closed under conjugations and finite intersections.

The symbol $\mathcal{O}_{\mathcal{D}}$ denotes *the orbit category* which has as objects transitive left G -sets with isotropy in \mathcal{D} and as morphisms G -equivariant maps, see [2]. Objects of the category $\mathcal{O}_{\mathcal{D}}$ have the form G/H where $H \in \mathcal{D}$.

A (right) $\mathcal{O}_{\mathcal{D}}$ -module \underline{M} is a contravariant functor on the category of orbits $\mathcal{O}_{\mathcal{D}}$ with values in the category of abelian groups. Such a module is determined by the abelian groups $\underline{M}(G/H)$ where $H \in \mathcal{D}$, and by a group homomorphism

$$\underline{M}(G/H) \rightarrow \underline{M}(G/H')$$

associated with any G -equivariant map $G/H' \rightarrow G/H$.

Let X be a G -CW-complex such that the isotropy subgroup of every point $x \in X$ belongs to the family \mathcal{D} . For every subgroup $H \in \mathcal{D}$ we may consider the cell complex X^H of H -fixed points and its cellular chain complex $C_*(X^H)$. A G -map $\phi : G/K \rightarrow G/L$ induces a cellular map $X^L \rightarrow X^K$ by mapping $x \in X^L$ to $gx \in X^K$ where g is determined by the equation $\phi(K) = gL$ (thus $g^{-1}Kgx = x$ since $g^{-1}Kg \subset L$ and therefore $Kgx = gx$, i.e. $gx \in X^K$). Thus we see that the chain complexes $C_*(X^H)$, considered for all $H \in \mathcal{D}$, form a chain complex of right $\mathcal{O}_{\mathcal{D}}$ -modules which will be denoted $\underline{C}_*(X)$; here

$$\underline{C}_*(X)(G/H) = C_*(X^H).$$

Note that the complex $\underline{C}_*(X)$ is free as a complex of $\mathcal{O}_{\mathcal{D}}$ -modules although the complex $C_*(X)$ might not be free.

There is an obvious augmentation $\epsilon : \underline{C}_0(X) \rightarrow \mathbb{Z}$ which reduces to the usual augmentation $C_0(X^H) \rightarrow \mathbb{Z}$ on each subgroup $H \in \mathcal{D}$.

If \underline{M} is a right $\mathcal{O}_{\mathcal{D}}$ -module, we may consider the cochain complex of $\mathcal{O}_{\mathcal{D}}$ -morphisms $\text{Hom}_{\mathcal{O}_{\mathcal{D}}}(\underline{C}_*(X), \underline{M})$. Its cohomology

$$(15) \quad H_{\mathcal{D}}^*(X; \underline{M}) = H^*(\text{Hom}_{\mathcal{O}_{\mathcal{D}}}(\underline{C}_*(X), \underline{M}))$$

is *the Bredon equivariant cohomology of X with coefficients in \underline{M}* .

Let M denote the principal component of \underline{M} . Reducing to the principal components we obtain a homomorphism of cochain complexes

$$\text{Hom}_{\mathcal{O}_{\mathcal{D}}}(\underline{C}_*(X), \underline{M}) \rightarrow \text{Hom}_{\mathbb{Z}[G]}(C_*(X), M)$$

and the homomorphism on the cohomology groups

$$(16) \quad H_{\mathcal{D}}^i(X; \underline{M}) \rightarrow H_G^i(X, M).$$

5.2. If the action of G on X is free then the homomorphism (16) is an isomorphism and

$$H_{\mathcal{D}}^i(X; \underline{M}) \simeq H^i(X/G, M)$$

where on the right we have the usual twisted cohomology. In particular we obtain

$$H_{\mathcal{D}}^n(E(\pi^r), \underline{M}) = H^n(\pi^r, M).$$

5.3. Suppose now that $X = E(\pi^{r-1})$, viewed as a left G -CW-complex, where $G = \pi^r$, see §4.5. We know that $E(\pi^{r-1})$ is a model for the classifying space $E_{\mathcal{D}}(G)$ and the classifying complex $E_{\mathcal{D}}(G)$ is unique up to G -homotopy. Hence we may use the notation

$$H_{\mathcal{D}}^*(E(\pi^{r-1}), \underline{M}) = H_{\mathcal{D}}^*(\pi^r, \underline{M}).$$

We obtain that the number $\text{cd}_{\mathcal{D}}(\pi^r)$ coincides with the maximal integer n such that

$$H_{\mathcal{D}}^i(\pi^r, \underline{M}) = 0$$

for all $i > n$ and for all $\mathcal{O}_{\mathcal{D}}$ -modules \underline{M} .

5.4. Consider now the effect of the π^r -equivariant map $F : E(\pi^r) \rightarrow E(\pi^{r-1})$, see (8). Note that any two equivariant maps $E(\pi^r) \rightarrow E(\pi^{r-1})$ are equivariantly homotopic. The induced map on Bredon cohomology

$$F^* : H_{\mathcal{D}}^i(E(\pi^{r-1}), \underline{M}) \rightarrow H_{\mathcal{D}}^i(E(\pi^r), \underline{M})$$

together with the notations introduced in §5.2 and §5.3 produce a homomorphism

$$(17) \quad \Phi : H_{\mathcal{D}}^i(\pi^r, \underline{M}) \rightarrow H^i(\pi^r, M)$$

which relates the Bredon cohomology with the usual group cohomology of π^r .

Now we state a result which gives useful lower bounds for the topological complexity $\text{TC}_r(X)$.

Theorem 5.1. *Let X be a finite aspherical cell complex with fundamental group π . Suppose that for some $\mathcal{O}_{\mathcal{D}}$ -module \underline{M} there exists a Bredon cohomology class*

$$\underline{\alpha} \in H_{\mathcal{D}}^n(\pi^r, \underline{M})$$

such that the class

$$\Phi(\underline{\alpha}) \neq 0 \in H^n(\pi^r, M)$$

is nonzero. Here M denotes the principal component of \underline{M} . Then $\text{TC}_r(X) \geq n$.

Proof. Suppose that $\text{TC}(X) < n$. Then by Theorem 3.1 the map $F : E(\pi^r) \rightarrow E_{\mathcal{D}}(\pi^r)$ admits a factorisation

$$E(\pi^r) \rightarrow L \rightarrow E_{\mathcal{D}}(\pi^r)$$

where L is a G -CW-complex of dimension less than n . Then the homomorphism

$$\Phi : H_{\mathcal{D}}^n(\pi^r, \underline{M}) \rightarrow H^n(\pi^r, M)$$

factorises as

$$\Phi : H_{\mathcal{D}}^n(\pi^r, \underline{M}) \rightarrow H_{\mathcal{D}}^n(L, \underline{M}) \rightarrow H^n(\pi^r, M)$$

and the middle group vanishes since $\dim L < n$. This contradicts our assumption that $\Phi(\underline{\alpha}) \neq 0$ for some $\underline{\alpha} \in H_{\mathcal{D}}^n(\pi^r, \underline{M})$. \square

6. PROOF OF THEOREM 2.2: HIGMAN'S GROUP

G. Higman gave an example of a 4-generator, 4-relator group with some remarkable properties. First, form the group H_{xy} with presentation

$$\langle x, y \mid xyx^{-1}y^{-2} \rangle.$$

This group is isomorphic to the Baumslag-Solitar group $B(1, 2)$, and hence is a duality group of dimension 2.

The infinite cyclic group $F(y)$ injects into both H_{xy} and H_{yz} , and so we may form

$$H_{xyz} := H_{xy} *_{F(y)} H_{yz}.$$

We may also form H_{zwx} as the amalgamated sum of H_{zw} and H_{wx} over $F(w)$. The free group $F(x, z)$ injects into both H_{xyz} and H_{zwx} , and Higman's group is defined to be

$$\mathcal{H} := H_{xyz} *_{F(x, z)} H_{zwx}.$$

It has presentation

$$\mathcal{H} = \langle x, y, z, w \mid xyx^{-1}y^{-2}, yzy^{-1}z^{-2}, zwz^{-1}w^{-2}, wwx^{-1}x^{-2} \rangle.$$

The group \mathcal{H} is acyclic (it has the same integer homology as a trivial group), and so $\tilde{H}^*(\mathcal{H}; k) = 0$ for every abelian group k . Moreover, it has no non-trivial finite dimensional representations over any field and so if M is any coefficient $\mathbb{Z}[\mathcal{H}]$ -module which is finitely generated as an abelian group, then $\tilde{H}^*(\mathcal{H}; M) = 0$. Thus the group \mathcal{H} is difficult to distinguish from a trivial group using cohomological invariants. On the other hand, since \mathcal{H} is not a free group so we have $\text{cd}(\mathcal{H}) \geq 2$. The 2-dimensional complex associated to the presentation of \mathcal{H} given above is aspherical and it follows that

$$\text{cat}(\mathcal{H}) = \text{cd}(\mathcal{H}) = \dim(K(\mathcal{H}, 1)) = 2$$

where, by \dim we refer to the smallest dimension of a $K(H, 1)$ complex. Thus the higher topological complexity of Higman's group satisfies $\text{TC}_r(\mathcal{H}) \leq 2r$, using the general result that

$$\text{TC}_r(X) \leq \text{cat}(X^r) \leq \dim(X^r) \leq r \dim(X).$$

Note that the zero-divisors cup length over any field is zero, so lower cohomological bounds are hard to come by. In [18] using a geometric group theory argument due to Yves de Cornulier, it was shown that $gH_{xy}g^{-1} \cap H_{zw} = \{1\}$ for all $g \in \mathcal{H}$. Furthermore, both of these groups are isomorphic to the Baumslag-Solitar group $B(1, 2)$, hence are duality groups of dimension 2. Thus the product

$$K = H_{xy} \times H_{zw} \times \cdots \times H_{zw}$$

(with $r - 1$ factors H_{zw}) is a duality group of dimension $2r$ and so $\text{cd}(H_{xy} \times H_{zw}^{r-1}) = 2r$. We may apply Theorem 2.1 to obtain

$$2r = \text{cd}(H_{xy} \times H_{zw}^{r-1}) \leq \text{TC}_r(\mathcal{H}) \leq 2r$$

so that $\text{TC}_r(\mathcal{H}) = 2r$. □

7. THE HIGHER TOPOLOGICAL COMPLEXITY OF RIGHT ANGLED ARTIN GROUPS

7.1. Let $\Gamma = (V, E)$ be a finite graph and let $H = H_\Gamma$ be the right angled Artin (RAA) group associated to Γ . Recall that $H = H_\Gamma$ is given by a presentation with generators $v \in V$ and relations $vw = wv$, for each edge $(v, w) \in E$. In Theorem 7.2 below we state the result of [17] which computes the topological complexity $\text{TC}_r(H_\Gamma)$. Our goal here is to give a new vastly simplified proof of the relevant lower bound using Theorem 2.1. An upper bound implying the equality requires finding explicit motion planners and this may be found in [17]. We shall need the following definition.

Definition 7.1. For a graph $\Gamma = (V, E)$ and for an integer $r \geq 2$ we define the number $z_r(\Gamma)$ as the maximal total cardinality $\sum_{i=1}^r |C_i|$ of r cliques $C_1, \dots, C_r \subset V$ with empty intersection, $\cap_{i=1}^r C_i = \emptyset$.

Recall that a clique of a graph $\Gamma = (V, E)$ is a set of vertices $C \subset V$ such that any two are connected by an edge. In other words, a clique is a complete induced subgraph of Γ .

One may equivalently define $z_r(\Gamma) = \max \{ \sum_{i=1}^r |C_i| - |\cap_{i=1}^r C_i| \}$ where C_1, \dots, C_r run over all sequences of r cliques in Γ . Since our original definition is included in this one the only question is whether the new definition can give a strictly greater number. To see that this cannot happen we note that given an arbitrary sequence of r cliques C_1, \dots, C_r we may modify it by subtracting from the last clique C_r the intersection $\cap_{i=1}^r C_i$ obtaining a sequence as in the original definition with the same value of the total sum.

Theorem 7.2. [17] One has $\text{TC}_r(H_\Gamma) = z_r(\Gamma)$.

In [17] the main result is stated slightly differently since the authors operate in higher generality and use a different language. However it easy to see that Theorem 7.2 follows from Theorem 2.7 and Proposition 2.3 in [17]. Below we shall see that the lower bound $z_r(\Gamma) \leq \text{TC}_r(H_\Gamma)$ follows directly and easily from simple results about cliques and Theorem 2.1.

We first observe that, if $c(\Gamma)$ denotes the size of the maximal clique, then

$$(18) \quad (r-1)c(\Gamma) \leq z_r(\Gamma) \leq rc(\Gamma).$$

The right inequality follows from $|C_i| \leq c(\Gamma)$. The right inequality can be strict if the graph Γ contains r cliques of maximal size $c(\Gamma)$ with disjoint intersection. To prove the left inequality we note that we may always take $C_1 = \dots = C_{r-1}$ of size $c(\Gamma)$ and $C_r = \emptyset$. The estimates given by (18) are in fact the algebraic analogue of the topological estimates in (4).

7.2. Let $A \subset V$ be a subset. We shall denote by $[A] \subset H = H_\Gamma$ the subgroup generated by A . We shall also denote by $A^\perp \subset H$ the normal subgroup generated by the set $V - A$. Note that we do not exclude the case $A = \emptyset$; in that case $[A] = 1$ and $A^\perp = H$.

7.3. Every subset $A \subset V$ determines a homomorphism $f_A : H \rightarrow H$ as follows. We define f_A on the set of generators $V \subset H$ by setting $f_A(v) = v$ for $v \in A$ and $f_A(v) = 1$ for $v \in V - A$. For every relation $vw = wv$ of H , either (1) both vertices $v, w \in A$ are in A ,

or (2) only one of the vertices v, w lies in A , or (3) none of v, w lies in A . In either case we have $f_A(v)f_A(w) = f_A(w)f_A(v)$, which shows that the homomorphism f_A is well defined.

For any two subsets $A, B \subset V$ one has

$$f_A \circ f_B = f_{A \cap B} = f_B \circ f_A,$$

and $f_\emptyset = 1$, $f_V = \text{id}_H$. Extending multiplicatively, the image of f_A coincides with $[A]$ and moreover $f_A(x) = x$ for any $x \in [A]$. In particular, if A is a clique, then $\text{Im}(f_A) = [A]$ is a free abelian group on the vertices in A .

Lemma 7.3. *For any two subsets $A, B \subset V$ one has $[A] \cap [B] = [A \cap B]$.*

Proof. Obviously $[A \cap B] \subset [A] \cap [B]$; hence we only need to show that $[A] \cap [B] \subset [A \cap B]$. If $x \in [A] \cap [B]$ then $f_{A \cap B}(x) = f_A(f_B(x)) = x$ implying that $x \in \text{Im}(f_{A \cap B}) = [A \cap B]$. \square

Proposition 7.4. *For any set of cliques $C_1, C_2, \dots, C_r \subset V$ with empty intersection, $\cap_{i=1}^r C_i = \emptyset$, one has*

$$\bigcap_{i=1}^r (g_i[C_i]g_i^{-1}) = \{1\},$$

for any collection of elements $g_1, \dots, g_r \in H$.

Proof. Let $x \in \bigcap_{i=1}^r (g_i[C_i]g_i^{-1})$. Then $x = g_i c_i g_i^{-1}$ with $c_i \in [C_i]$ for $i = 1, \dots, r$. Then, for all i and j , we have $c_j = g_j^{-1} g_i c_i (g_j^{-1} g_i)^{-1}$ so that, by the discussion above, we have $f_{C_i}(c_j) = c_i$ since each $[C_i] = \text{Im}(f_{C_i})$ is abelian. But then applying the equality $f_{C_i}(c_j) = c_i$ inductively r times, we obtain $f_{C_1} \circ f_{C_2} \circ \dots \circ f_{C_r}(c_1) = c_1$. Hence using the fact that

$$f_{C_1} \circ f_{C_2} \circ \dots \circ f_{C_r} = f_{\cap_{i=1}^r C_i}$$

has image equal to $[\cap_{i=1}^r C_i] = \{1\}$ we see that $c_1 = 1$ and hence $x = 1$. \square

Corollary 7.5. *Let $C_1, \dots, C_r \subset V$ be a set of cliques with empty intersection. Then the group*

$$K = [C_1] \times [C_2] \times \dots \times [C_r] \subset H^r$$

satisfies the condition of Theorem 2.1, i.e. for any subgroup $L \subset H^r = H \times H \times \dots \times H$ which is conjugate to the diagonal $\Delta \subset H^r$ one has $L \cap K = \{1\}$.

We have now shown that $z_r(\Gamma)$ is given by a certain set of cliques C_1, \dots, C_r with empty total intersection and these in turn determine a subgroup $K = [C_1] \times [C_2] \times \dots \times [C_r] \subset H^r$ with

$$\text{cd}(K) = \sum_{i=1}^r |C_i| = z_r(\Gamma)$$

since each $[C_i]$ is free abelian. Applying Theorem 2.1 then provides the lower bound $z_r(\Gamma) \leq \text{TC}_r(H_\Gamma)$.

8. THE TC-GENERATING FUNCTION

8.1. For a group H consider the following TC-generating function:

$$(19) \quad \mathcal{F}_H(x) = \sum_{r=1}^{\infty} \text{TC}_{r+1}(H) \cdot x^r.$$

It is a formal power series whose coefficients are the integers $\text{TC}_r(H)$.

Example 8.1. Let \mathcal{H} be the Higman's group as in Theorem 2.2. Then $\text{TC}_r(\mathcal{H}) = 2r$ for any r and the TC-generating function has the form

$$\mathcal{F}_{\mathcal{H}}(x) = \sum_{r=1}^{\infty} 2(r+1)x^r = \frac{2x(2-x)}{(1-x)^2}.$$

In this section we make the following observation:

Theorem 8.2. *Let $H = H_{\Gamma}$ be a right angled Artin group. Then the TC-generating function $\mathcal{F}_H(x)$ is a rational function of the form*

$$\mathcal{F}_H(x) = \frac{P_{\Gamma}(x)}{(1-x)^2},$$

where $P_{\Gamma}(x)$ is an integer polynomial with $P_{\Gamma}(1) = c(\Gamma) = \text{cd}(H_{\Gamma})$.

The proof of Theorem 8.2 given below uses the following lemmas in which we assume that $H = H_{\Gamma}$ is a RAA group associated to a graph $\Gamma = (V, E)$. We abbreviate the notation $z_r(\Gamma)$ to z_r .

Lemma 8.3. *Suppose that $C_1, \dots, C_r \subset V$ is a sequence of cliques such that $\cap_{i=1}^r C_i = \emptyset$ and $\sum_{i=1}^r |C_i| = z_r$. If additionally $\cap_{i=1}^{r-1} C_i = \emptyset$ then $z_r = z_{r-1} + c(\Gamma)$.*

Proof. Note that $z_{r-1} = \sum_{i=1}^{r-1} |C_i|$ and $|C_r| = c(\Gamma)$ since otherwise we would be able to increase the sum $\sum_{i=1}^r |C_i|$ by replacing C_1, \dots, C_{r-1} by a sequence realising z_{r-1} and by replacing C_r by a clique of size $c(\Gamma)$. The result follows. \square

Next we consider the case when r is large enough.

Lemma 8.4. *For $r > n = |V|$ one has $z_r = z_{r-1} + c(\Gamma)$.*

Proof. Let $C_1, \dots, C_r \subset V$ be a sequence of cliques with $\cap_{i=1}^r C_i = \emptyset$ and $\sum_{i=1}^r |C_i| = z_r$. Our statement will follow from Corollary 8.3 once we know that the intersection of some $r-1$ cliques out of C_1, \dots, C_r is empty. Suppose the contrary, i.e. for any fixed $j = 1, 2, \dots, r$ the intersection $\cap_{i \neq j} C_i \neq \emptyset$. Then we can find a point $x_j \in \cap_{i \neq j} C_i$, i.e. $x_j \in C_i$ for any $i \neq j$. Clearly $x_j \notin C_j$ since the total intersection of the cliques C_i is empty. We obtain a sequence of points $x_1, \dots, x_r \in V$ and from our construction it is obvious that they are all pairwise distinct. But this contradicts our assumption $r > n = |V|$. \square

Proof of Theorem 8.2. Using Lemma 8.4, by induction we find

$$(20) \quad z_r = z_n + (r - n)c(\Gamma) \quad \text{for } r \geq n.$$

Using the equation $\mathrm{TC}_r(H_\Gamma) = z_r(\Gamma)$ given by Theorem 7.2 we see that

$$\begin{aligned} \mathcal{F}_H(x) &= \sum_{r=1}^{n-1} z_{r+1}x^r + \sum_{r=n}^{\infty} z_{r+1}x^r \\ &= \sum_{r=1}^{n-1} z_{r+1}x^r + (z_n - nc(\Gamma)) \sum_{r=n}^{\infty} x^r + c(\Gamma) \cdot \sum_{r=n}^{\infty} (r+1)x^r \\ &= \sum_{r=1}^{n-1} z_{r+1}x^r + (z_n - nc(\Gamma)) \frac{x^n}{1-x} + \frac{c(\Gamma)}{(1-x)^2} - c(\Gamma) \sum_{r=0}^{n-1} (r+1)x^r. \end{aligned}$$

The result follows since the first and the fourth terms are integer polynomials and the second and the third terms can be written as rational functions with denominator $(1-x)^2$. \square

For an RAA group H_Γ , it is known that the maximum size $c(\Gamma)$ of a clique is equal to the cohomological dimension of H_Γ (which, in fact, is also the LS category of $K(H_\Gamma, 1)$). Hence, we obtain the following.

Theorem 8.5. *If H_Γ is an RAA group and $|V_\Gamma| = n$, then*

$$\mathrm{TC}_r(H_\Gamma) = \mathrm{TC}_n(H_\Gamma) + (r-n)\mathrm{cd}(H_\Gamma)$$

for $r \geq n$.

This is interesting because, while $\mathrm{TC}_r(X) - \mathrm{TC}_{r-1}(X) \leq 2\mathrm{cat}(X)$ holds for any X (see [1], Proposition 3.7), we see that for RAA groups we have a precise description of the difference in terms of homological information about the group,

$$\mathrm{TC}_r(H_\Gamma) - \mathrm{TC}_{r-1}(H_\Gamma) = \mathrm{cd}(H_\Gamma) = \mathrm{cat}(K(H_\Gamma, 1))$$

for $r > n$.

Example 8.6. Suppose that $H = H_\Gamma$ is a free group on $n \geq 2$ generators. In this case the graph Γ has n vertices and no edges. We see that $z_r(\Gamma) = r$ for all $r \geq 2$. Hence

$$\mathcal{F}_H(x) = \frac{x(2-x)}{(1-x)^2}.$$

Example 8.7. In the other extreme, suppose that Γ is a complete graph on n vertices. Then $H = \mathbb{Z}^n$ and $z_r(\Gamma) = (r-1)n$ for all $r \geq 2$. We obtain

$$\mathcal{F}_H(x) = \frac{nx}{(1-x)^2}.$$

8.2. Naturally, one may ask if the phenomenon of Theorem 8.2 holds in greater generality. More specifically, we ask *for which finite CW-complexes X the formal power series*

$$\mathcal{F}_X(x) = \sum_{r=1}^{\infty} \mathrm{TC}_{r+1}(X) \cdot x^r$$

represents a rational function of the form

$$\frac{P_X(x)}{(1-x)^2}$$

where $P_X(x)$ is an integer polynomial satisfying

$$P_X(1) = \text{cat}(X)?$$

The question above is equivalent to the statement that for all r large enough the following recurrence relation

$$\text{TC}_{r+1}(X) = \text{TC}_r(X) + \text{cat}(X)$$

holds. It would be interesting to know the answer in the case when $X = K(G, 1)$ for various classes of groups, say, for the class of hyperbolic groups.

Next we consider the following examples.

Example 8.8. If $X = S^{2k+1}$, then we know that $\text{TC}_r(X) = r - 1$. Therefore, we have

$$\mathcal{F}_X(x) = x \cdot \sum_{r=0}^{\infty} (r+1) \cdot x^r = \frac{x}{(1-x)^2}.$$

For the even-dimensional sphere $X = S^{2k}$ we have $\text{TC}_r(S^{2k}) = r$ and

$$\mathcal{F}_X(x) = \frac{x(2-x)}{(1-x)^2}.$$

Example 8.9. If $X = G$ a compact Lie group, then we know that $\text{TC}_r(G) = \text{cat}(G^{r-1})$. Thus

$$\mathcal{F}_X(x) = \sum_{r=1}^{\infty} \text{cat}(G^r) \cdot x^r.$$

Let's take a specific example where we know $\text{cat}(G)$. Let $G = U(n)$. Then we know that $\text{cat}(U(n)) = \text{cup}_{\mathbf{Q}}(U(n)) = n$ where $\text{cup}_{\mathbf{Q}}$ denotes the rational cuplength. Note that rational cuplength obeys $\text{cup}_{\mathbf{Q}}(X^r) = r \cdot \text{cup}_{\mathbf{Q}}(X)$ for any X . Hence we have $\text{cat}(U(n)^r) = rn$ from the following:

$$rn = r \cdot \text{cup}_{\mathbf{Q}}(X) = \text{cup}_{\mathbf{Q}}(X^r) \leq \text{cat}(U(n)^r) \leq r \cdot \text{cat}(U(n)) = rn.$$

We then have

$$\mathcal{F}_{U(n)}(x) = \sum_{r=1}^{\infty} rn x^r = \frac{nx}{(1-x)^2}.$$

Note that $\mathcal{F}_{U(n)}(x)$ coincides with the TC-generating function for the n -dimensional torus $\mathcal{F}_{T^n}(x)$, see example 8.7.

Example 8.10. Let M^{2n} be a closed simply connected symplectic manifold. Then $\text{TC}_r(M) = rn$ for any $r = 1, 2, \dots$, see [1], Corollary 3.15. Therefore,

$$\mathcal{F}_M(x) = \frac{nx(2-x)}{(1-x)^2}.$$

Example 8.11. Let Σ_g denote the orientable surface of genus $g \geq 2$. Then, since $\mathrm{TC}_r(\Sigma_g) = 2r$ (see [16, Proposition 3.2]), we obtain similarly to Example 8.1,

$$\mathcal{F}_{\Sigma_g}(x) = \frac{2x(2-x)}{(1-x)^2}.$$

In the cases $g = 0$ and $g = 1$ the answers are different, see Example 8.7 and Example 8.8.

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